TA Notes on Week 5
Macroeconomics III - TI1716

Alessandro Zona Mattioli

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In these notes I provide some more details on the derivations of the Hopenhayn (1992) model and Melitz (2003) model done in week 5 of the course Macro III at the Tinbergen Institute. I am grateful to anyone who can spot typos or suggest improvements.

1 Hopenhayn (1992)

The Hopenhayn model is a quite theoretical model, with the advantage of being very general and able to encapsulate multiple functional forms and model extensions. Indeed, these features made it the workhorse framework for most of the following literature on firm dynamics.

1.1 Model Ingredients

- A continuum of firms uniquely characterized by heterogeneous productivity $\phi \in S := [0, 1]$. ϕ is independent across firms and evolves according to the Markov process $F(\phi'|\phi)$.
- Firms produce a homogeneous good q with the unique input n, so that the production process is summarized by $q = f(\phi, n)$.
- Upstream, the input supply $W(N)$ is assumed non decreasing in aggregate labour N.
- Downstream, market demand $p = D(Q)$ is strictly decreasing in aggregate quantity Q.
- Firms behave competitively, so they take prices for the final good p and intermediate input w as given. Moreover, to stay on the market they pay a fixed fee c_f at every period.
- The mix of the parameters $\{\phi, w, p\}$ determine the profit, output supply, and input demand functions: $\pi(\phi, w, p)$, $q(\phi, w, p)$, $n(\phi, w, p)$. All these functions are assumed to be increasing in ϕ.

So far the model is quite standard, the main novelty is introduced via **entry and exit** dynamics. We indeed assume that firms in the market (the incumbents) have the possibility to exit, while there is a potentially infinite amount of firms ready to enter the market (the entrants). Moreover, there is free entry in this economy. How the process works is modeled in the following way:

1.1.1 Timing

1. (a) Incumbents decide whether to stay in the market or exit.
   (b) Entrants decide whether to enter paying the fixed fee c_e or not.
2. (a) Incumbents see the realization of $\phi' \sim F(\phi'|\phi)$.
   (b) Entrants are hit by an initial productivity shock $\nu \sim G(.)$.
3. Firms make output decisions and prices adjust so to equate aggregate demand and supply.

It follows that in each period some firms will find themselves with a productivity level that does not allow them to make enough profits to survive, therefore they will exit from the market in the next period. Similarly, entrants will enter as long as the expected profits from participating in the economy are non negative. Notice that, of course, expected profits are a function of productivity. Since we are in a competitive economy, the more firms enter, the less profits are there to share, so entry will take place until expected profits are equal to 0.

It is worth stressing that the dynamics in the model are solely determined by productivity, which is the only stochastic process. A key role is taken by the fixed fees c_f, c_e, which implicitly determine a dynamic cutoff-productivity level that moderates the processes of entry and exit.

The realization of the shocks at each period and the resulting number of firms are summarized by $M_t = \mu_t(S)$, or the "mass" of firms in the market. We can finally aggregate over such measure and retrieve aggregate supply of the final good as well as aggregate demand for labour, respectively:

$$\begin{aligned} Q^s(\mu, p, w) &= \int q(\phi, w, p)\mu(d\phi) \\ N^d(\mu, p, w) &= \int n(\phi, w, p)\mu(d\phi) \end{aligned} \tag{1}$$

Starting from an initial level of μ_0, the equilibrium will be determined by the sequence of $\{p_t, w_t, \mu_t(\phi_t)\}$. Notice that the first two are fully deterministic, while the third is stochastic.

1.2 Equilibrium

Since the economy is competitive, in every period w and p adjust to balance the final good market and the input market, therefore:

$$\begin{aligned} w_t^* &: N_t^d = W(N_t) \; \forall t \\ p_t^* &: Q_t^s = D(Q_t) \; \forall t \end{aligned} \tag{2}$$

Graphically:

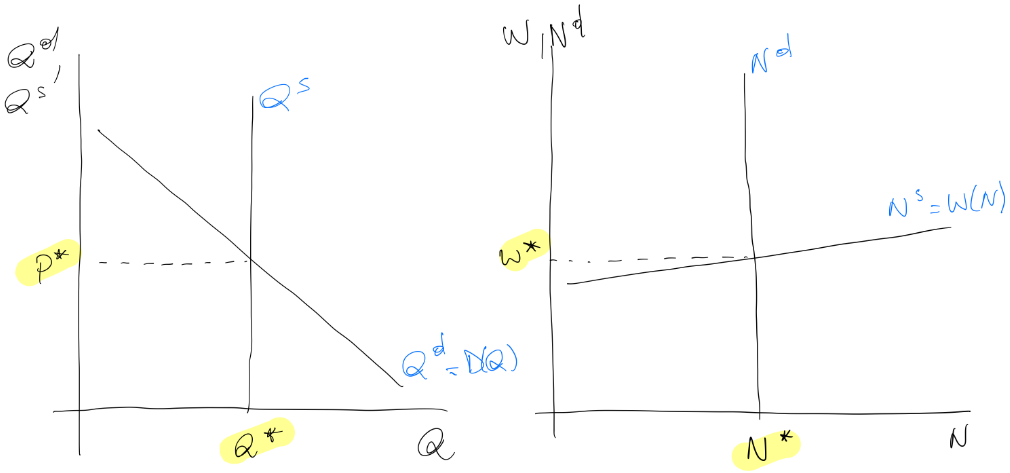

Figure 1: Determination of equilibrium w and p.

On the other hand, the entry and exit graphical representation is represented in the figure below:

Figure 2: Productivity cutoff for entry and exit.

More formally, the model derivations boil down to the determination of the firm problem, from the point of view of the entrants and of the incumbents.

1.2.1 Incumbent

We can represent the value function for the incumbent firm at time t_0 using a Bellman equation in the following way:

$$v_t(\phi, \{p_t, w_t\}_{t_0}^{\infty}) = v_t(\phi, z) = max\{\pi(\phi, p_t, w_t)\} + \beta max\left\{0, \int v_{t+1}(\phi', z)F(d\phi'|\phi)\right\} \quad (3)$$

where $z = \{p_t, w_t\}_{t_0}^{\infty}$ denotes the future stream of equilibrium prices and β is the firm discount factor. The firm therefore maximizes current period profits and decides whether it will exit in the next period, collecting payoff 0, or it will stay, maximizing the expected value of the next period $v(.)$, integrated over the possible realizations of the Markov process for productivity.

The second term of the value function implicitly defines a cutoff productivity level below which firms will exit in the next period (ϕ^* in Figure 2):

$$\phi^* = x_t = inf\left\{\phi \in S \text{ s.t. } \int v_{t+1}(\phi', z)F(d\phi'|\phi) \geq 0\right\} \quad (4)$$

where x_t is according to the original notation of the paper.

1.2.2 Entrant

The entrant, on the other hand, will evaluate the expected profits from entering against the outside option of staying out of the market and earning 0 profit:

$$v_t^e(z) = \int v_t(\phi, z)\nu(d\phi) \quad (5)$$

where the expected payoff depends on the realization of the initial productivity shock ν. As already argued above, the free entry condition will imply that firms will keep on entering as long as it is profitable, therefore in equilibrium we can derive that: $v_t^e(z) = c_e$. So expected value from entering will be just enough to compensate the entry fee c_e.

The two entry/exit rules imply a law of motion for μ_t such that, for every $\phi' \in [0, 1]$:

$$\mu_{t+1}([0, \phi']) = \int_{\phi \geq x_t} F(\phi'|\phi)\mu_t(d\phi) + M_{t+1}G(\phi') \quad (6)$$

Where the first term is the population of incumbents whose period t productivity lies above x_t and $t+1$ productivity is ϕ', while the second term is the mass of new entrants whose productivity lies below ϕ'. This is a compact and quite convenient representation of the dynamics of μ_t and is useful to define the equilibrium.

Finally, the competitive equilibrium is a set of sequences $\{p_t^*\}$, $\{w_t^*\}$, $\{Q_t^*\}$, $\{N_t^*\}$, $\{M_t^*\}$, $\{x_t^*\}$, $\{\mu_t^*\}$ s.t. in each period:

- p_t^*, w_t^* are such that (2) is satisfied.
- x_t^* is such that (4) is satisfied.
- $v_t^e(z) = c_e$ whenever the mass of entrants $M_t > 0$.
- μ_t^* is defined recursively by (6), given μ_0, M_t^*, x_t^*.

This formulation of the equilibrium, yet still quite implicit, offers a general setup for formulating and solving more explicit versions of the model. Moreover, the equilibrium conditions imply an order of solving that can be adopted either on paper or on the computer. In the problem set you might be asked to solve a simplified version of such setup, similar to the one laid out in the next subsection.

1.3 Toy Model

In this section I will adapt the notation to that of the slides, so slightly differentiating from the one of the previous section. Yet I will try to point out the parallelisms between the two models.

1.3.1 Main ingredients

- There is still a continuum of firms characterized by productivity $a_t \sim G(.)$. Differently from the main model though, firms receive a productivity draw only *once*, when they enter, so there is no Markov process and productivity is fixed for the incumbents.
- Firms produce a homogeneous good y according to the production function $y = a(l - c_f)^\alpha$, where l is the unique input and c_f are overhead costs.
- Upstream input supply $W(N)$ and output demand $p = D^{-1}(Q)$ have the same properties as in the main model.
- Firms behave competitively, there is no fixed cost for staying in the market (its "role" is taken by c_f), but firms are subject to an exogenous job destruction shock that arrives at rate δ.
- Similarly to the main model, the mix of the parameters $\{a, w, p\}$ determine the profit, output supply, and input demand functions: $\pi(a, w, p)$, $y(a, w, p)$, $l(a, w, p)$. All these functions are assumed to be increasing in a.

The mechanics of the model are thereafter similar as in the main model, with the important distinction that exit is now exogenously determined by the job destruction shock, so we can focus just on entry decisions. The resulting dynamics are:

1.3.2 Timing

1. Entrants decide whether to enter paying the fixed fee c_e.
2. - Incumbents exit the market with probability δ.
   - Entrants draw their productivity a_t from $G(.)$.
3. Firms make output decisions and prices adjust so to equate aggregate demand and supply.

In this case we simply denote the mass of incumbents as N_t and that of entrants as M_t, for each period. We can then derive the aggregate variables similarly to the main model (with a slightly lighter notation):

$$
\begin{aligned}
Q^s(p) &= \int y(a,p)dG(a) = \frac{N}{1-G(a_d)}\int_{a_d} y(a,p)dG(a) \\
L^d(w) &= \int l(a,w)dG(a) = \frac{N}{1-G(a_d)}\int_{a_d} l(a,w)dG(a)
\end{aligned}
\tag{7}
$$

Where a_d is the productivity cutoff for entry, that we will derive in the next subsection. Notice that one can also directly adjust the definition of $G()$, changing it with a truncated CDF: $G_{a_d}()$, so that the above quantities can be rewritten as:

$$
\begin{aligned}
Q^s(p) &= \int y(a,p)dG(a) = N\int_{a_d} y(a,p)dG_{a_d}(a) \\
L^d(w) &= \int l(a,w)dG(a) = N\int_{a_d} l(a,w)dG_{a_d}(a)
\end{aligned}
\tag{8}
$$

Finally, as in the main model, we assume free entry.

1.3.3 Equilibrium

The definition of the equilibrium mainly boils down to the entrant firm problem. Its optimization problem can be summarized with the value function:

$$
\begin{aligned}
v_t^e(a,p) &= \sum_{t=0}^{\infty}\left(\frac{1-\delta}{1+r}\right)^t E(\pi^*(a,p)) \\
&= \frac{1}{1-\frac{1-\delta}{1+r}}\int \pi^*(a,p)dG(a) = \frac{1+r}{r+\delta}\int \pi^*(a,p)dG(a)
\end{aligned}
\tag{9}
$$

where $\pi^*()$ are the optimal profits. In this simpler model, the value of entering is simply the expected stream of expected profits, discounted by the discount rate and the risk of exiting the market exogenously. It is possible to notice that also in this simple model the entry condition implicitly defines a productivity threshold a_d below which profits will be negative and therefore firms will not enter the market. In other words, we can define a_d s.t.:

$$\pi(a_d, p) = 0 \tag{10}$$

and so we can rewrite (9) as:

$$v_t^e(a, p) = \frac{1+r}{r+\delta} \int_{a_d} \pi^*(a, p) dG(a) \tag{11}$$

Since we assume free entry, in equilibrium it must hold that $v_t^e(a, p) = c_e$, so in other words:

$$\int_{a_d} \pi^*(a, p) dG(a) = \frac{r+\delta}{1+r} c_e \tag{12}$$

It still remains to be determined $\pi^*()$. This is simply given by the solution to:

$$max_{l \geq 0}\ \pi(a, p) = py(a, l) - wl = pa(l - c_f)^\alpha - wl \tag{13}$$

so that taking FOC wrt l we can derive optimal labour demand:

$$\begin{aligned} \alpha pa(l - c_f)^{\alpha-1} &= w \\ l^*(p, a) &= \left(\frac{w}{\alpha pa}\right)^{\frac{1}{\alpha-1}} c_f \end{aligned} \tag{14}$$

and then we can plug back l^* into the production function to retrieve $y^*(p, a)$ and finally $\pi^*(p, a)$.

As a last step before defining the equilibrium, we notice that entry and exit flows will need to balance each other in equilibrium, therefore the number of incumbents that are forced to exit by the exogenous job destruction shock will need to balanced by the mass of new entrants. This is summarized by the equation below:

$$\delta N = M(1 - G(a_d)) \tag{15}$$

where the left hand side depicts exit flows and the right hand side entry flows of firms.

With this final condition we pin down all the required parameters, so the equilibrium will be given by a set of sequences $\{p_t^*\}, \{w_t^*\}, \{Q_t^*\}, \{M_t^*\}, \{N_t^*\}, \{L_t^*\}$ and a_d s.t. in every period:

- cutoff productivity condition:
$$\pi(a_d, p^*) = 0$$

- free entry:
$$\int_{a_d} \pi^*(a, p^*) dG(a) = \frac{r+\delta}{1+r} c_e$$

- $\{p_t^*\}, \{w_t^*\}$ are such that markets clear, as by (7).

- equilibrium in flows:
$$\delta N_t^* = M_t^*(1 - G(a_d))$$

2 Melitz (2003)

The Melitz model is an important breakthrough for International Economics and Macro, more in general. The model builds on Krugman (1980), which in turns was able to explain intra–industry trade with the help of *Dixit-Stiglitz* preferences (able to model "love of variety") and increasing returns to scale.
It shifted the focus from country-to-country analysis to a micro founded approach and was able to explain some important empirical facts, such as firm selection induced by trade and productivity growth induced by reallocation of resources towards more efficient firms. In this course we just sketch some of its main ingredients, but for more details I suggest taking International Economics in Block V.

2.1 Environment

The model is mainly composed of a demand side and a supply side:

- The demand side is populated by a representative consumer with Dixit-Stiglitz preferences over a continuum of varieties of a homogeneous good.

- The supply side is composed by a continuum of firms, each producing a single variety of the homogeneous good. This implies that each firm has monopoly power over its variety. Moreover, firms are subject to entry and exit, similarly to Hopenhayn (1992).

To start solving the model, we proceed according to the following steps:

1. Solve for demand side, taking prices as given.

2. Solve for supply side, taking demand as given.

2.2 Demand

The representative consumer solves the following utility maximization problem subject to a budget constraint:

$$\begin{aligned} max_{y(a)}\ U &= \left[\int_{a\in S} y(a)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \\ s.t.\ R &= \int_{a\in S} p(a)y(a)da = wL \end{aligned} \tag{16}$$

where $a \in S$ denote the single varieties, $y(a)$ and $p(a)$ quantity and price for variety a and R is aggregate expenditure over varieties, while wL is labour income, i.e. wage multiplied by hours worked. In what follows, we can assume labour supply L to be fixed: $L = \bar{L}$.

We can then solve for consumption (demand) and expenditure for each variety: $y(a)$ and $r(a) = p(a)y(a)$.

We start by setting up the Lagrangean:

$$\mathcal{L} = \left[\int_{a\in S} y(a)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} + \lambda\left[wL - \int_{a\in S} p(a)y(a)da\right] \tag{17}$$

Then we take FOC wrt $y(a)$ and rearrange:

$$\begin{aligned} \frac{\sigma}{\sigma-1}\left[\int_{a\in S} y(a)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} y(a)^{-\frac{1}{\sigma}} &= \lambda p(a) \\ y(a)^{-\frac{1}{\sigma}}\left[\int_{a\in S} y(a)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} &= \lambda p(a) \end{aligned} \tag{18}$$

We can now take the ratio of demand for variety a over demand for variety b:

$$\begin{aligned} \left(\frac{y(a)}{y(b)}\right)^{-\frac{1}{\sigma}} &= \frac{p(a)}{p(b)} \\ y(a) &= y(b)\left(\frac{p(a)}{p(b)}\right)^{-\sigma} \end{aligned} \tag{19}$$

In a possibly surprising step, we then multiply both sides by $p(b)^{1-\sigma}$ and integrate over b:

$$
\begin{aligned}
\int p(b)^{1-\sigma} y(a) db &= \int p(b)^{1-\sigma} y(b) \left(\frac{p(a)}{p(b)}\right)^{-\sigma} db \\
y(a) \int p(b)^{1-\sigma} db &= p(a)^{-\sigma} \int p(b) y(b) db \\
y(a) P^{1-\sigma} &= p(a)^{-\sigma} R \\
y^*(a) &= \frac{R}{P} \left(\frac{p(a)}{P}\right)^{-\sigma} = Y \left(\frac{p(a)}{P}\right)^{-\sigma}
\end{aligned}
\tag{20}
$$

where a key step is possible after defining the price index $P = \left[\int p(a)^{1-\sigma} da\right]^{\frac{1}{1-\sigma}}$ and noticing that $P^{-1}R = Y$ ($= Q$ in the slides).

It is then straight forward to define expenditure per variety as:

$$
r^*(a) = p(a) y^*(a) = R \left(\frac{p(a)}{P}\right)^{1-\sigma} \tag{21}
$$

2.3 Supply

Once we have obtained the equilibrium demand and expenditure given the price level, we can now move to the supply side: solve for equilibrium price $p(a)$ taking $y(a)$ as given.

Differently from the Krugman (1980) model, here firms are heterogeneous for what concerns productivity ϕ. Now, before any confusion, notice that each firm produces a unique variety a and is characterized by the unique productivity level ϕ, therefore the notation might sometimes use ϕ and a interchangeably. For now, we stick to denoting firms by their productivity level.

Firms produce using only labour according to the input demand: $l(y, \phi) = y/\phi$, so for a given level of output, higher productivity allows to require less labour. Workers need to be paid wages w; moreover firms need to face a fixed cost of production c_f. Therefore we can set up firms' maximization problem as:

$$
\begin{aligned}
max_{p(a)} \pi(\phi) = r^*(\phi) - l(\phi) w - c_f &= p(\phi) y^*(\phi) - \frac{y^*(\phi)}{\phi} w - c_f \\
&= p(\phi)^{1-\sigma} \frac{Y}{P^{-\sigma}} - \frac{w}{\phi} \left(\frac{p(\phi)}{P}\right)^{-\sigma} Y - c_f
\end{aligned}
\tag{22}
$$

where in the second line we plug in equilibrium demand. So taking the FOC wrt $p(\phi)$ and rearranging:

$$
\begin{aligned}
(1-\sigma)\left(\frac{p(\phi)}{P}\right)^{-\sigma} Y + \frac{\sigma}{\phi}\frac{Y}{P^{-\sigma}}p(\phi)^{-\sigma-1} &= 0 \\
\frac{1-\sigma}{\sigma}p(\phi) &= -\frac{w}{\phi} \\
p(\phi) &= \frac{\sigma}{\sigma-1}\frac{w}{\phi} = \frac{1}{\rho}\frac{w}{\phi}
\end{aligned}
\tag{23}
$$

where ρ denotes the markup the firm is able to impose on marginal costs, thanks to its market power.

It is now possible to rewrite profits in the following way:

$$
\begin{aligned}
\pi(\phi) = r(\phi) - l(\phi)w - c_f &= r(\phi) - \frac{y(\phi)}{\phi}w - c_f \\
&= r(\phi) - \frac{r(\phi)}{p(\phi)\phi}w - c_f \\
&= r(\phi)\left(1 - \frac{w}{\phi}\frac{\sigma-1}{\sigma}\frac{\phi}{w}\right) - c_f \\
&= \frac{r(\phi}{\sigma} - c_f
\end{aligned}
\tag{24}
$$

Moreover, from the demand side it follows that:

$$
\begin{aligned}
r(\phi) = R\left(\frac{p(\phi)}{P}\right)^{1-\sigma} &= \frac{R}{P^{1-\sigma}}\left(\frac{\sigma}{\sigma-1}\frac{w}{\phi}\right)^{1-\sigma} = R\left(P\rho\frac{\phi}{w}\right)^{\sigma-1} \\
\implies \pi(\phi) &= \frac{R}{\sigma}\left(P\rho\frac{\phi}{w}\right)^{\sigma-1} - c_f \\
\implies l(\phi) &= y(\phi)\frac{w}{\phi} + c_f = \frac{wY}{\phi}\left(\frac{w}{\rho\phi P}\right)^{-\sigma} + c_f = Y(\rho P)^{\sigma}\left(\frac{w}{\phi}\right)^{1-\sigma} + c_f
\end{aligned}
\tag{25}
$$

From the above we can retrieve:

$$
\begin{aligned}
\frac{r(a)}{l(a)} &= \frac{R\left(P\rho\frac{a}{w}\right)^{\sigma-1}}{Y(\rho P)^{\sigma}\left(\frac{w}{a}\right)^{1-\sigma} + c_f} \\
&= \frac{1}{\rho}\frac{Y(\rho P)^{\sigma}\left(\frac{w}{a}\right)^{1-\sigma}}{Y(\rho P)^{\sigma}\left(\frac{w}{a}\right)^{1-\sigma} + c_f} = \frac{1}{\rho}\left[1 - \frac{c_f}{l(a)}\right]
\end{aligned}
\tag{26}
$$

so that higher productivity leads to higher measured revenue productivity. Additionally, we can see that more productive firms enjoy higher revenues and produce more:

$$
\begin{aligned}
\frac{y(a_1)}{y(a_2)} &= \frac{Y\left(\frac{w}{\rho a_1 P}\right)^{-\sigma}}{Y\left(\frac{w}{\rho a_2 P}\right)^{-\sigma}} = \left(\frac{a_1}{a_2}\right)^{\sigma} \\
\frac{r(a_1)}{r(a_2)} &= \frac{R\left(\frac{w}{\rho a_1 P}\right)^{1-\sigma}}{R\left(\frac{w}{\rho a_2 P}\right)^{1-\sigma}} = \left(\frac{a_1}{a_2}\right)^{\sigma-1}
\end{aligned}
\tag{27}
$$

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