# TA Notes on Week 6
# Macroeconomics III - TI1716

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In these notes I provide some more details on the derivations of Restuccia Rogerson model (2007), done in week 6 of the course Macro III at the Tinbergen Institute. I present derivations to support results contained in the slides, but for more details do refer to the paper themselves. I am grateful to anyone who can spot typos or suggest improvements.

## 1 Restuccia Rogerson (2007)

The model is fairly simple in its analytical components. It builds on the structure of Hopenhayn (1992), but it introduces few differences:

- Instead of using a general "Demand side", this model relies on a proper consumer problem, as we have a representative household characterized by the standard utility function $u(C_t)$, that owns capital and allocates labour. The household also owns the heterogeneous production plants and allocates aggregate capital and labour to them via a fixed supply.
- Plants on the other hand, are heterogeneous in productivity and face the same entry and exit dynamics as in Hopenhayn (1992), plus:
  - an exogenous exit probability (similarly to the Toy model);
  - when firms enter in the market they not only draw their productivity level, but also the distortion they are subject to, which can be positive, negative or neutral. This is modeled via a draw from a bivariate distribution.

These are the main characteristics of the model, which is standard for what concerns the rest. The derivation of its equilibrium then boils down to solving the consumer problem and the entry problem at the plant level. The rest is aggregation.

### 1.1 Consumer problem

The consumer solves the following utility maximization problem:

$$
\begin{aligned}
max_{\ C_t, K_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u(C_t) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} p_t(C_t + K_{t+1} - (1-\delta)K_t) = \sum_{t=0}^{\infty} p_t(r_t K_t + w_t N_t + \Pi_t - T_t) \\
& K_0 \text{ given}
\end{aligned}
\tag{1}
$$

where $p_t$ is the time zero price of period $t$ consumption, $w_t$ and $r_t$ are the period $t$ rental prices of labor and capital, $\Pi_t$ is the total profit from the operations of all plants, and $T_t$ is the lump-sum taxes levied by the government. $N_t$ is total labor services supplied to the market, which will always be equal to one since the individual does not value leisure. Notice that the problem already incorporates the evolution of capital and investment. You can immediately see that the problem is quite simple, indeed by writing the Lagrangean and taking the FOC wrt $C_t$ and $K_{t+1}$:

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(C_t) + \lambda_t \cdot p_t \cdot [r_t K_t + w_t N_t + \Pi_t - T_t - C_t - K_{t+1} + (1-\delta)K_t] \right\} \\
FOC_{C_t}: \ u'(C_t) = \lambda_t \cdot p_t \\
FOC_{K_{t+1}}: \ \lambda_t p_t = \lambda_{t+1} \cdot p_{t+1} \cdot \beta \cdot (r_{t+1} + (1-\delta)) \\
\implies u'(C_t) = u'(C_{t+1}) \cdot \beta \cdot (r_{t+1} + (1-\delta))
\end{aligned}
\tag{2}
$$

therefore a solution with a constant $C_t$ and $r_t$ (or a fixed demand side, as in Hopenhayn (1992)) requires that:

$$
r = \frac{1}{\beta} - (1-\delta) \tag{3}
$$

Then the authors define the corresponding interest rate:

$$
R = r - \delta = \frac{1}{\beta} - 1 \tag{4}
$$

## 1.2 Incumbent Firm problem

The firm, once entered, will face each period a static profit maximization problem, since input decisions are not intertemporal. So an incumbent firm will choose to enter or not (binary decision $\bar{x} \in \{0, 1\}$) based on the discounted stream of these optimal profits. the problem is expressed as:

$$W_e = \int_{(s,\tau)} max_{\bar{x}}[W(s,\tau)]dG(s,\tau) - c_e \tag{5}$$

Where $c_e$ is the cost of entry and $W(s,\tau)$ is the discounted static problem:

$$W(s,\tau) = \frac{\pi(s,\tau}{1-\rho} \tag{6}$$

with $\rho = \frac{1-\lambda}{1+R}$, with $\lambda$ being the exogenous exit rate.

Notice that differently from Hopenhayn (1992), here the firms face a joint distribution of $s$ and $\tau$, so upon entry they will draw both their productivity level and their distortion $\tau$. Therefore, in case we wanted to bring the model to a computer, we would have to specify a *bivariate* distribution $dG(s,\tau)$, where $G(.)$ is a CDF.

We can now derive the optimal input demand for $k$ and $l$ by solving the period $t$ profit maximization:

$$\begin{aligned} \pi(s,\tau) &= max_{n,k}\ (1-\tau)sk^{\alpha}n^{\gamma} - wn - rk - c_f \\ FOC_k:&\ \ \alpha(1-\tau)sk^{\alpha-1}n^{\gamma} = r \\ FOC_n:&\ \ \gamma(1-\tau)sk^{\alpha}n^{\gamma-1} = w \end{aligned} \tag{7}$$

Taking the ratio of the two FOC we obtain:

$$\begin{aligned} \frac{\alpha}{\gamma}\frac{n}{k} &= \frac{r}{w} \\ n &= k\frac{r}{w}\frac{\gamma}{\alpha} \\ k &= n\frac{w}{r}\frac{\alpha}{\gamma} \end{aligned} \tag{8}$$

where the last two lines are just rewriting the first. Then we can plug the second line in the FOC for $k$ and the third in the FOC for $n$ to obtain:

$$\begin{aligned} \bar{k}(s,\tau) &= \left(\frac{\alpha}{r}\right)^{\frac{1-\gamma}{1-\gamma-\alpha}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma-\alpha}} (s(1-\tau))^{\frac{1}{1-\gamma-\alpha}} \\ \bar{n}(s,\tau) &= \left(\frac{(1-\tau)s\gamma}{w}\right)^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{1-\gamma}} \end{aligned} \tag{9}$$

### 1.3 Aggregation

Aggregation follows the lines of Hopenhayn (1992), as the invariant distribution of firms is defined as:

$$\mu'(s,\tau) = (1-\lambda)\mu(s,\tau) + \bar{x}(s,\tau)dG(s,\tau)E \tag{10}$$

and $E$ is the mass of potential entrants in each period. We can see that as long as the share of firms opting for entry $\bar{x}(s,\tau) \in (0,1)$ this is a contraction mapping and therefore has a unique fixed point $\hat{\mu}(s,\tau)$, which is simply the solution to the equation above:

$$\hat{\mu}(s,\tau) = \frac{\bar{x}(s,\tau)}{\lambda}dG(s,\tau) \tag{11}$$

Finally, we can obtain equilibrium if we impose on top of these conditions the familiar:

- Free entry: $W_e = 0$;
- Market clearing:

$$\begin{aligned} \text{Labour market } L^s &= \int_{s,\tau} \bar{n}(s,\tau)d\mu(s,\tau) \\ \text{Capital market } K &= \int_{s,\tau} \bar{k}(s,\tau)d\mu(s,\tau) \\ \text{Aggregate costs } C + \delta K + c_e E &= \int_{s,\tau} (f(s,k,\tau) - c_f)d\mu(s,\tau) \end{aligned} \tag{12}$$

- Balanced budget:

$$T + \int_{s,\tau} \tau f(s,k,\tau)d\mu(s,\tau) = 0 \tag{13}$$

# References

Hopenhayn, H. A. (1992). Entry, exit, and firm dynamics in long run equilibrium. *Econometrica: Journal of the Econometric Society*, 1127–1150.

Restuccia, D., & Rogerson, R. (2008). Policy distortions and aggregate productivity with heterogeneous establishments. *Review of Economic dynamics*, *11*(4), 707–720.