TA Notes on Week 6
Macroeconomics III - TI1716

Alessandro Zona Mattioli

April 2021

In these notes I provide some more details on the derivations of part of the basic model in Hopenhayn (2014) and Hsieh & Klenow (2009), done in week 6 of the course Macro III at the Tinbergen Institute. I present derivations to support results contained in the slides, but for more details do refer to the paper themselves. I am grateful to anyone who can spot typos or suggest improvements.

1 Hopenhayn (2014)

1.1 Perfect Competition

In the Basic Model at the beginning of the paper, the economy is assumed to have a *fixed* number of firms M, which compete according to perfect competition and produce according to the production function:

$$y_i = z_i n_i^{\eta} \ , \ 0 < \eta < 1 \tag{1}$$

where n_i is the only input of production and z_i is firm specific productivity.

The optimization problem faced by the planner is:

$$\begin{aligned} max_{n_i} \ & \sum_i z_i n_i^{\eta} \\ s.t. \ & \sum_i n_i \leq N \end{aligned} \tag{2}$$

We can therefore lay down the Lagrangean and take FOC wrt n_i:

$$
\begin{aligned}
\mathcal{L} &= \sum_i z_i n_i^{\eta} + \lambda(N - \sum_i n_i) \\
FOC_{n_i} &: \ \eta z_i n_i^{\eta-1} = \lambda \ \forall i
\end{aligned} \tag{3}
$$

Therefore the optimal condition for the planner is to equate marginal product of the unique production input across firms.

Moreover, if we take the ratio between the FOC of two general firms i and j we obtain:

$$
\begin{aligned}
\frac{\eta z_i n_i^{\eta-1}}{\eta z_j n_j^{\eta-1}} &= 1 \\
\frac{z_i n_i^{\eta}/n_i}{z_j n_j^{\eta}/n_j} &= 1 \\
\frac{y_i/n_i}{y_j/n_j} &= 1
\end{aligned} \tag{4}
$$

from which we can see that the average product of all firms is equal to the same ratio, or as in the paper: $y_i/n_i = y/n = a$. Therefore we can derive that in aggregate:

$$
\begin{aligned}
&\frac{y_i}{n_i} = a \implies \frac{\sum_i y_i}{n_i} = \sum_i a \implies \frac{My}{N} = Ma \\
&\frac{y}{N} = a
\end{aligned} \tag{5}
$$

Moreover, we can express optimal demand for the intermediate input as:

$$
\begin{aligned}
n_i^* &= \frac{y_i}{a} = \frac{z_i n_i^{\eta}}{a} \\
&= \left(\frac{z_i}{a}\right)^{\frac{1}{1-\eta}}
\end{aligned} \tag{6}
$$

Therefore, by using the expression for n_i^*, we can conclude that aggregate production y can be expressed as:

$$
\begin{aligned}
y &= \sum_i z_i n_i^\eta = \sum_i z_i \left(\frac{z_i}{a}\right)^{\frac{1}{1-\eta}} = \frac{\sum_i z_i^{\frac{1}{1-\eta}}}{a^{\frac{\eta}{1-\eta}}} \\
\left(\frac{y}{N}\right)^{\frac{\eta}{1-\eta}} y &= \sum_i z_i^{\frac{1}{1-\eta}} \\
\frac{y}{N^\eta} &= \left(\sum_i z_i^{\frac{1}{1-\eta}}\right)^{1-\eta} \\
y &= \left(\sum_i z_i^{\frac{1}{1-\eta}}\right)^{1-\eta} N^\eta
\end{aligned}
\tag{7}
$$

Moreover, we can divide and multiply the right hand side of aggregate production by $M^{1-\eta}$:

$$
y = \left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{M}\right)^{1-\eta} M^{1-\eta} N^\eta = \left(E z_i^{\frac{1}{1-\eta}}\right)^{1-\eta} M^{1-\eta} N^\eta \tag{8}
$$

So that we can see that production exhibits decreasing returns both in the number of firms (M) and in aggregate labour (N), but constant returns in the aggregate ($1-\eta < 1$, $\eta < 1$, $\eta + 1 - \eta = 1$). This is a mirror of the assumption of fixed M.

1.2 Monopolistic Competition

If we introduce monopolistic competition, as for instance in Melitz (2003), aggregate production has slightly different characteristics. Production takes place in two steps, 1) a downstream representative firm that operates in perfect competition and produces according to the CES aggregator:

$$
y = \left[\int y_i^\eta di\right]^{\frac{1}{\eta}} \tag{9}
$$

and 2) a continuum of monopolists of the varieties $\{y_i\}$, which produce according to:

$$
y_i = \tilde{z}_i n_i \tag{10}
$$

where similarly to before $\tilde{z}_i$ is productivity and n_i is the unique input. After some derivations (see notes on Melitz model) we can conclude that aggregate production is of the type:

$$
y = \left(E z_i^{\frac{1}{1-\eta}}\right)^{\frac{1-\eta}{\eta}} M^{\frac{1-\eta}{\eta}} N \propto \left(E \tilde{z}_i^{\frac{\eta}{1-\eta}}\right)^{\frac{1-\eta}{\eta}} M^{\frac{1-\eta}{\eta}} N \tag{11}
$$

where, if M is again fixed, then aggregate production and productivity are as in the perfect competition case, but this time we have increasing returns to scale.

1.3 Entry & Extensive Margin

If we move to a framework similar to Hopenhayn (1992) or its simplified version (Toy model), then we can derive yet different properties for aggregate production. We assume that the fixed entry cost c_e is expressed in terms of workers and we assume free entry.

It follows that the optimization problem for the firm is given by:

$$E\pi(w) = \int \pi(z,w)G(dz)\ ,\ \pi = max_n zn^\eta - wn \tag{12}$$

where $G()$ is the usual CDF for productivity. Taking the FOC wrt n in the profit maximization we obtain:

$$\begin{aligned} \eta z n^{\eta-1} &= w \\ n &= \left(\frac{\eta z}{w}\right)^{\frac{1}{1-\eta}} \end{aligned} \tag{13}$$

and finally free entry implies: $E\pi(w) = wc_e$. Then the problem for the planner that wants to set optimal levels of M and N_e, so respectively new entrants and incumbents, can be set as:

$$\begin{aligned} max_{M,N_e}\ & AM^{1-\eta}N_e^\eta \\ s.t.\ & c_eM + N_e \leq N \end{aligned} \tag{14}$$

where we assume an aggregate Cobb-Douglas production function and the planner faces an aggregate resource constraint. From the FOC we derive:

$$\begin{aligned} FOC_{N_e}\ & \eta AM^{1-\eta}N_e^{\eta-1} = \lambda \\ FOC_M\ & (1-\eta)AM^{-\eta}N_e^\eta = \lambda c_e \\ \implies\ & \frac{\eta}{1-\eta}\frac{1}{c_e}M = N_e \end{aligned} \tag{15}$$

Together with the budget constraint these conditions allow us to finally retrieve:

$$\begin{aligned} N_e &= \eta N \\ M &= \frac{(1-\eta)N}{c_e} \\ w &= \lambda \text{ (by definition)} \end{aligned} \tag{16}$$

In conclusion, the aggregate production function becomes:

$$y = A_0 c_e^{-(1-\eta)} N \tag{17}$$

which this time intuitively exhibits decreasing returns in c_e and constant returns in N.

In all these models we assume optimal allocation of resources and inputs, i.e. equalization of marginal returns across firms. Nonetheless, there is extensive literature that shows how this can also not be the case. We saw a first example of this in the financial accelerator model (Bernanke et al., 1999), where financial frictions prevented firms from achieving optimal size. The next model presents yet another theory of this phenomenon.

2 Hsieh & Klenow (2009)

In what follows, I will try to sketch some of the derivation needed for the Hsieh & Klenow (2009) paper. I will follow he notation of the paper, but it should be straight forward to use these notes also for the slides.

This study tries to estimate the extent of misallocation in China and India, compared to the US. The authors focus on manufacturing and try to quantify the dispersion in marginal products of labour and capital across plants. Before turning to the empirical analysis, they construct the following model. There are three sectors:

- Single final good Y, produced by a representative firm in a perfectly competitive market. It purchases S different inputs from an upstream intermediate input sector. Cobb-Douglas production technology is assumed.
- Intermediate input Y_s, which aggregates (or assembles) a set of M_s varieties $\{Y_{si}\}_i$ produced upstream. They produce according to a CES-aggregator technology.
- Further upstream intermediate input Y_{si} produced using K_{si}, L_{si} and A_{si} according to a Cobb-Douglas CRS production technology. This sector faces general frictions or distortions:
  - markup on relative cost of capital wrt labour: $\tau_{K_s i}$.
  - tax on the price of goods $\tau_{Y_{si}}$.

By solving this 3-sectors economy model, we can derive an empirical test to measure the extent of misallocation.

2.1 Final good sector

Final good producers maximize total profits $PY - \sum P_s Y_s$. Production is Cobb-Douglas, so $Y = \prod_s Y_s^{\theta_s}$, where θ_s is the elasticity of each input. P_s is the price of input Y_s and

$P = \prod_s (P_s/\theta_s)^{\theta_s}$ is the final good price. For tractability and without loss of generality, we can impose $P = 1$. We can therefore set up the maximization problem as:

$$max\ _{Y_s} PY - \sum P_s Y_s = max\ _{Y_s} P \prod_s Y_s^{\theta_s} - \sum P_s Y_s \tag{18}$$

So taking the FOC wrt Y_s:

$$\frac{\theta_s}{Y_s} \prod_s Y_s^{\theta_s} = P_s \implies P_s Y_s = \theta_s PY \tag{19}$$

We can then move to the next sector upstream, which will face demand Y_s.

2.2 Intermediate good sector s

Each intermediate good s is produced using M_s inputs according to the production function:

$$Y_s = \left[\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{20}$$

Therefore, if P_{si} is the price of each intermediate input Y_{si}, we can set up a similar maximization problem as before. We first optimize wrt Y_{si} and then derive the optimal price P_s that the firm will charge to downstream final good producers:

$$max\ _{Y_{si}} P_s Y_s - \sum_{i=1}^{M_s} P_{si} Y_{si} = max\ _{Y_{si}} P_s \left[\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - \sum_{i=1}^{M_s} P_{si} Y_{si} \tag{21}$$

So we can go ahead and take FOC wrt Y_{si} and obtain:

$$\begin{aligned} \frac{\sigma}{\sigma-1} P_s \left[\sum_{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma}} \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} &= P_{si} \\ P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}} &= P_{si} \\ P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}} &= P_{si} Y_{si} \end{aligned} \tag{22}$$

where the last line is convenient for the following steps.

We will now move to P_s, proving that it must be of the type:

$$P_s = \left[\sum_{i=1}^{M_s} P_{si}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{23}$$

This is used also in Melitz (2003) and other examples in this literature, so it is useful in general to understand where it comes from. But you can also trust this result and go straight to the next subsection.

This is essentially the result of a cost minimization problem, symmetric to the profit maximization problem we just discussed:

$$P_s Y_s = min\ _{Y_{si}} \sum_{M_s} P_{si} Y_{si} \ \text{ s.t. } \ Y_s = \left[\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{24}$$

We therefore set up a Lagrangean, rearranging a bit the constraint:

$$\mathcal{L} = \sum_{M_s} P_{si} Y_{si} + \lambda_s \left[Y_s^{\frac{\sigma}{\sigma-1}} - \sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right] \tag{25}$$

So taking FOC:

$$\begin{aligned} P_{si} - \lambda_s \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} &= 0 \\ P_{si} = \lambda_s \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} & \end{aligned} \tag{26}$$

We can plug this into total expenditure and derive:

$$\sum_{i=1}^{M_s} P_{si} Y_{si} = \sum_{i=1}^{M_s} \lambda_s \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} Y_{si} = \lambda_s \frac{\sigma-1}{\sigma} \sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} = \lambda_s \frac{\sigma-1}{\sigma} Y_s^{\frac{\sigma-1}{\sigma}} \tag{27}$$

Moreover, we can plug the FOC in and solve the third line of equation (22) for Y_{si}:

$$Y_{si}^{\frac{\sigma-1}{\sigma}} = \left[\frac{1}{P_{si}} \lambda_s \frac{\sigma-1}{\sigma} \right]^{\sigma-1} \tag{28}$$

Then plugging the above in the constraint of the cost minimization problem we obtain:

$$Y_s = \left[\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} = \left(\lambda_s \frac{\sigma-1}{\sigma} \right)^{\sigma} \left[\sum_{i=1}^{M_s} \frac{1}{P_{si}}^{\sigma-1} \right]^{\frac{\sigma}{\sigma-1}} \tag{29}$$

which allows us to solve for λ_s:

$$\lambda_s = \frac{\sigma}{\sigma - 1} Y_s^{\frac{1}{\sigma}} \left[\sum_{i=1}^{M_s} \frac{1}{P_{si}}^{\sigma - 1} \right]^{\frac{-1}{\sigma - 1}} \tag{30}$$

We can then revise the formula for total expenditure (27) by plugging in the expression for λ_s:

$$\begin{aligned} \sum_{i=1}^{M_s} P_{si} Y_{si} &= \lambda_s \frac{\sigma - 1}{\sigma} Y_s^{\frac{\sigma - 1}{\sigma}} \\ &= Y_s^{\frac{1}{\sigma}} \left[\sum_{i=1}^{M_s} \frac{1}{P_{si}}^{\sigma - 1} \right]^{\frac{-1}{\sigma - 1}} Y_s^{\frac{\sigma - 1}{\sigma}} \\ &= Y_s \left[\sum_{i=1}^{M_s} P_{si}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \end{aligned} \tag{31}$$

So from the objective function of the cost minimization problem we can finally conclude that:

$$P_s = \left[\sum_{i=1}^{M_s} P_{si}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{32}$$

This basically means that each unit of Y_s costs P_s if each intermediate input Y_{si} is purchased in the efficient proportion (i.e. as it is determined by the parameter σ). This index will therefore show up in all cases of CES-aggregator production function and is therefore used as numeraire in many models. Notice that both intermediate good producer s and si enjoy monopoly power over their good, but producer s is able to fully pass through the increase in costs to final demand, as it is monopolist itself.

We can now finally move to the last sector.

2.3 Further intermediate good sector si

As a recap, for each sector S there are M_s producers of the intermediate inputs $\{Y_{si}\}$. Each of those produce according to the following (more familiar) Cobb-Douglas production function:

$$Y_{si} = A_{si} K_{si}^{\alpha_s} L_{si}^{1-\alpha_s} \tag{33}$$

Notice that productivity A_{si} is firm specific, while the production technology is sector specific (the coefficients are α_s and not α_{si}). This is required to make the model match with the data.

As we said in the beginning, there are two firm specific distortions $\tau_{K_s i}$ and $\tau_{Y_{si}}$. These imply that that the profit function is of this type:

$$\pi_{si} = (1 - \tau_{Y_{si}}) P_{si} Y_{si} - w L_{si} - (1 + \tau_{K_s i}) R K_{si} \tag{34}$$

where w and R are the input cost of L_{si} and K_{si}, respectively. We can now set up the maximization problem, by plugging in the expression for $P_{si}Y_{si}$ we derived from the past subsection in (22), which is taken as given by these firms:

$$\begin{aligned} max_{K_{si}, L_{si}} \pi_{si} &= (1 - \tau_{Y_{si}}) P_{si} Y_{si} - w L_{si} - (1 + \tau_{K_s i}) R K_{si} \\ &= (1 - \tau_{Y_{si}}) P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}} - w L_{si} - (1 + \tau_{K_s i}) R K_{si} \end{aligned} \tag{35}$$

So we can take the FOC wrt K_{si} and L_{si} to obtain:

$$\begin{aligned} (1 - \tau_{Y_{si}}) P_s Y_s^{\frac{1}{\sigma}} \frac{\sigma - 1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} \alpha_s A_{si} \left(\frac{L_{si}}{K_{si}} \right)^{1-\alpha_s} &= (1 + \tau_{K_s i}) R \\ (1 - \tau_{Y_{si}}) P_s Y_s^{\frac{1}{\sigma}} \frac{\sigma - 1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} (1 - \alpha_s) A_{si} \left(\frac{L_{si}}{K_{si}} \right)^{-\alpha_s} &= w \end{aligned} \tag{36}$$

If we take the ratio of the two conditions we can retrieve:

$$\left(\frac{L_{si}}{K_{si}} \right) = \frac{1 - \alpha_s}{\alpha_s} \frac{(1 + \tau_{K_s i}) R}{w} \tag{37}$$

Finally, we plug this ratio into the FOC for L_{si} and use the expression of P_{si} from the second line of (22) to conclude:

$$\begin{aligned} (1 - \tau_{Y_{si}}) P_s Y_s^{\frac{1}{\sigma}} \frac{\sigma - 1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} (1 - \alpha_s) A_{si} \left(\frac{1 - \alpha_s}{\alpha_s} \frac{(1 + \tau_{K_s i}) R}{w} \right)^{-\alpha_s} &= w \\ \implies P_{si} = \frac{\sigma}{\sigma - 1} \left(\frac{R}{\alpha_s} \right)^{\alpha_s} \left(\frac{w}{1 - \alpha_s} \right)^{1-\alpha_s} \frac{(1 + \tau_{K_s i})^{\alpha_s}}{A_{si} (1 - \tau_{Y_{si}})} & \end{aligned} \tag{38}$$

which, similarly to what we saw with the Melitz model, implies that the firm charges a markup $\mu = \sigma/(\sigma - 1)$ over its marginal costs, thanks to the monopoly power over input Y_{si}.

2.4 TFPR and TFPQ

At this stage, the authors define $TFPQ_{si}$ and $TFPR_{si}$, where the first is firm level productivity and the second firm level revenue productivity:

$$
\begin{aligned}
TFPQ_{si} &= A_{si} = \frac{Y_{si}}{K_{si}^{\alpha_s} L_{si}^{1-\alpha_s}} \\
TFPR_{si} &= P_{si} A_{si} = \frac{P_{si} Y_{si}}{K_{si}^{\alpha_s} L_{si}^{1-\alpha_s}}
\end{aligned}
\tag{39}
$$

The distinction is important for measurability, as $TFPR$ can be measured with industry level deflators, while $TFPQ$ would require firm level prices (harder to find).

The authors notice that these are proportional to the marginal revenue product of labour and of capital. The marginal revenue products are given by the products of actual marginal products and marginal revenue. Therefore, in this setup they can be expressed as the derivatives of revenue net of taxes with respect to the two inputs and are defined by the authors in this way:

$$
\begin{aligned}
MRPL_{si} &= (1-\alpha_s) \frac{\sigma - 1}{\sigma} \frac{P_{si} Y_{si}}{L_{si}} = \frac{w}{1 - \tau_{Y_{si}}} \\
MRPK_{si} &= \alpha_s \frac{\sigma - 1}{\sigma} \frac{P_{si} Y_{si}}{K_{si}} = R \frac{1 + \tau_{K_{si}}}{1 - \tau_{Y_{si}}}
\end{aligned}
\tag{40}
$$

Using these definitions, we can show that firm level TFPR is proportional to a geometric average of the firm's marginal revenue products of capital and labor:

$$
\begin{aligned}
(MPRK_{si})^{\alpha_s} (MPRL)^{1-\alpha_s} &= \left(\alpha_s \frac{\sigma - 1}{\sigma} \frac{P_{si} Y_{si}}{K_{si}} \right)^{\alpha_s} \left(\alpha_s \frac{\sigma - 1}{\sigma} \frac{P_{si} Y_{si}}{K_{si}} \right)^{1-\alpha_s} \\
&= \alpha_s^{\alpha_s} (1-\alpha_s)^{1-\alpha_s} \frac{\sigma - 1}{\sigma} \frac{P_{si} Y_{si}}{K_{si}^{\alpha_s} L_{si}^{1-\alpha_s}} \\
&= w^{1-\alpha_s} R^{\alpha_s} \frac{1 + \tau_{K_{si}}}{1 - \tau_{Y_{si}}}
\end{aligned}
\tag{41}
$$

In the expression above it is indeed possible to notice that:

- $\alpha_s^{\alpha_s} (1-\alpha_s)^{1-\alpha_s} \frac{\sigma-1}{\sigma}$ is constant at the sector level.
- $w^{1-\alpha_s} R^{\alpha_s}$ is also constant at the sector level

Therefore:

$$
\frac{P_{si} Y_{si}}{K_{si}^{\alpha_s} L_{si}^{1-\alpha_s}} = TFPR_{si} \propto (MPRK_{si})^{\alpha_s} (MPRL)^{1-\alpha_s} \propto \frac{1 + \tau_{K_{si}}}{1 - \tau_{Y_{si}}}
\tag{42}
$$

So this implies that as long as there is some form of misallocation in the economy, $TFPR$ will vary across firms, while it should be constant at the sector level if $\tau_{K_{si}} = \tau_{Y_{si}} = 0$. On the other hand, $TFPQ$ will vary regardless of frictions. This basic implication will be the main building block of the empirical strategy of the paper.

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